NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2739

NUMERICAL DETERMINATION OF INDICIAL LIFT AND MOMENT FUNCTIONS FOR A TWO-DIMENSIONAL SINKING AND PITCHING AIRFOIL AT MACH NUMBERS 0.5 AND 0.6

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Washington
July 1952

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SUMMARY

The indicial lift and moment functions are determined approximately for sinking and pitching motion at Mach numbers M of 0.5 and 0.6. These functions are determined from a knowledge of the existing oscillatory coefficients at the low reduced frequencies and from approximate expressions of these coefficients at the high reduced frequencies.

The beginning portion of the indicial lift function associated with an airfoil penetrating a sharp-edge gust in subsonic flow is evaluated by use of an exact method. By use of an approximate method for determining the remaining portion, the complete indicial gust function is determined for $M = 0.5$, $M = 0.6$, and $M = 0.7$.

All the indicial lift and moment functions are approximated by an exponential series; the coefficients which appear in the exponential approximations for each indicial function are tabulated for $M = 0.5$, $M = 0.6$, and $M = 0.7$.

INTRODUCTION

In the study of transient flows, two types of airfoil motions have had special significance - a harmonically oscillating airfoil and an airfoil experiencing a sudden change in angle of attack. The lift and moment functions for an airfoil experiencing a sudden change in angle of attack are commonly referred to as indicial lift and moment functions. In references 1 and 2 the indicial lift and moment functions of a two-dimensional airfoil for sinking and pitching motion in a subsonic compressible flow are presented for $M = 0.7$. The functions are determined from the reciprocal relations between the indicial functions and the oscillatory coefficients. Examination of the oscillatory coefficients calculated by Dietze, Schade, and Possio (see, for example, refs. 3, 4, and 5) indicates that data are not available over a sufficient range of reduced frequencies for the direct utilization of this method to determine the indicial lift functions at other subsonic Mach numbers.

The indicial functions for $M = 0.8$ have been evaluated by Lomax, Heaslet, and Sluder, in reference 6, employing an alternate method applicable for both subsonic and supersonic Mach numbers. The alternate method for subsonic Mach numbers is characterized by an exact, readily evaluated expression for the beginning portion of the indicial functions and an approximate expression for the remaining portion. The solution for the remaining portion of the indicial function is lengthy and tedious, and, consequently, an alternate approach is used in the present paper to calculate this portion of the indicial functions.

In the present paper indicial functions for $M = 0.5$ and $M = 0.6$ are obtained in approximate form for an extensive range of airfoil travel. The beginning portions of the indicial functions are calculated by the method of reference 6. The difficulties of calculating the remaining portions of the indicial functions by the method of reference 6 are avoided by approximating the missing portions of the oscillatory coefficients of references 3 and 4 so that the method of references 1 and 2 could be used. The approximation is made in terms of the initial parts of the indicial functions and the known parts of the oscillatory coefficients. An estimate of the reliability of the indicial functions is also presented.

During the last stage of preparing the paper, it was found that Timman, Van de Vooren, and Greidanus have computed the oscillatory coefficients in terms of known functions for Mach numbers 0.35, 0.5, 0.6, 0.7, and 0.8 and for reduced frequencies ranging from 0.1 to 3.0 (see ref. 7). A discussion of the effects of differences between the data of reference 7 and that of the present investigation on the results of the calculations is included.

SYMBOLS

s	distance traveled, half-chords
ω	angular frequency
V	forward velocity of airfoil
c	chord

x,z,ξ,η	Cartesian coordinates
k	reduced-frequency parameter, $\omega c/2V$
k_0	highest reduced frequency for which oscillatory coefficients are known
κk_0	value of reduced frequency where maximum error in approximated oscillatory coefficients occurs
ϵ_∞	error in oscillatory coefficients at infinite reduced frequency
ϵ_{max}	maximum error in oscillatory coefficients for region $\kappa > 1$
$L(s)$	section lift, positive in downward direction
$M(s)$	section pitching moment about quarter-chord point, positive when moment tends to depress trailing edge
ρ	density
h	amplitude of vertical or sinking displacement, positive downward, half-chords
θ	angle of pitch, positive when trailing edge is lower than leading edge
q	rate of change of angle of pitch with respect to distance traveled in half-chords, positive when trailing edge is falling with reference to leading edge
$k_1(s)$	indicial lift function for a two-dimensional airfoil experiencing a sudden change in vertical velocity
$m_1(s)$	indicial moment function for a two-dimensional airfoil experiencing a sudden change in vertical velocity where moment is taken about quarter-chord point
$k_{1_q}(s)$	indicial lift function for a two-dimensional airfoil experiencing a sudden change in pitching velocity about its leading edge

M	Mach number
C_p	pressure coefficient
$m_{1_q}(s)$	indicial moment function for a two-dimensional airfoil experiencing a sudden change in pitching velocity about its leading edge, moment taken about quarter-chord point
U	velocity of sharp-edge gust
$k_2(s)$	indicial lift function for a two-dimensional airfoil due to penetration of a sharp-edge gust
$F_c(k) + iG_c(k)$	complex compressible oscillatory lift derivative due to sinking velocity
$M(k) + iN(k)$	complex compressible oscillatory moment derivative due to sinking velocity
$F_{c_q}(k) + iG_{c_q}(k)$	complex compressible oscillatory lift derivative due to pitching velocity only
$M_q(k) + iN_q(k)$	complex compressible oscillatory moment derivative due to pitching velocity only

$f(k) = F_c(k) - F_c(\infty)$

$m(k) = M(k) - M(\infty)$

$f_q(k) = F_{c_q}(k) - F_{c_q}(\infty)$

$m_q(k) = M_q(k) - M_q(\infty)$

$\alpha_n, \beta_n, a_n, b_n$	constants in exponential approximations
A, B	constants defined by equations (13) and (14)

Subscript:

$3c/4$	for airfoil rotating about three-quarter-chord point

METHOD OF ANALYSIS

The indicial functions are determined in two parts. In the region $0 \leqq s \leqq \frac{2M}{1 + M}$, readily determined indicial functions given in reference 6 are used. For the region $s > \frac{2M}{1 + M}$, the relatively more difficult method of reference 6 is avoided by an alternate approach which utilizes the reciprocal relations between the indicial and the oscillatory functions given in references 1 and 2. Before these relations could be applied at $M = 0.5$ and $M = 0.6$, it was necessary to determine the unknown oscillatory coefficients at the high reduced frequencies, $k > 1$. Approximate expressions for these coefficients are obtainable by combining the existing data for the oscillatory coefficients at the low reduced frequencies, $0 \leqq k \leqq 1$, with the indicial functions for the region $0 \leqq s \leqq \frac{2M}{1 + M}$ through the use of reciprocal relations. The reciprocal relations are then used to obtain the remaining portion of the indicial functions.

The following procedure was used to determine the oscillatory coefficients at the high reduced frequencies: The reciprocal relation between the indicial lift function due to sinking velocity and the in-phase oscillatory lift coefficient is (see eq. (14) of ref. 1)

$$k_1(s) = F_c(\infty) + \frac{2}{\pi}\int_0^\infty \frac{f(k)\sin ks}{k}\,dk \qquad (s > 0) \quad \text{(1a)}$$

or alternately,

$$\int_{k_0}^\infty \frac{f(k)\sin ks}{k}\,dk = \frac{\pi}{2}\,k_1(s) - \frac{\pi}{2}\,F_c(\infty) - \int_0^{k_0} \frac{f(k)\sin ks}{k}\,dk \quad \text{(1b)}$$

where $f(k) = F_c(k) - F_c(\infty)$.

In equation (1b) values of $k_1(s)$ are known from the method of reference 6 for $0 \leqq s \leqq \frac{2M}{1 + M}$. The value of $F_c(\infty)$ which has been shown in reference 1 to correspond to the value of $k_1(s)$ at $s = 0$

is, consequently, also known. The oscillatory coefficient $f(k)$ is considered known for the range of reduced frequencies $0 \leqq k \leqq k_0$. Thus, the right-hand side of equation (1b) is known for values of $0 \leqq s \leqq \frac{2M}{1 + M}$. In the region $k_0 < k < \infty$, $f(k)$ is unknown and is to be determined.

A solution for $f(k)$ in the region $k_0 < k < \infty$ can be expressed by a finite convergent series with unknown coefficients. Definition of these coefficients is possible by a simultaneous solution of a set of algebraic equations. The set of equations is formulated by substituting the finite series for $f(k)$ (together with appropriate boundary conditions) in the integral on the left side of equation (1b). This integral is evaluated to obtain an algebraic expression in terms of the unknown series coefficients and the variable s. Each simultaneous equation in the set is then obtained by assigning to the variable s a value selected in the range where $k_1(s)$ is known. The number of these equations should correspond to the number of series coefficients to be determined. The set of equations is then solved simultaneously for the series coefficients in the approximation for $f(k)$.

The accuracy of the approximate solution for $f(k)$ depends on the range of s over which $k_1(s)$ is known, the value of k_0, and the number of terms in the series. The error introduced by a finite series for $f(k)$ is minimized by the fact that the oscillatory coefficient for the higher reduced frequencies is associated primarily with the initial portion of the indicial function which is known.

Once the approximate expression for $f(k)$ is determined at the high reduced frequencies, $k > 1$, this expression and the oscillatory coefficients at the low reduced frequencies, $0 \leqq k \leqq 1$, can be substituted into equation (1a) to determine $k_1(s)$ for values of $s > \frac{2M}{1 + M}$.

The same over-all method may be used for determining the indicial moment on a sinking airfoil as well as the indicial lift and moment functions for a pitching airfoil. The reciprocal relation used for approximating the oscillatory moment coefficients at the high reduced frequencies due to sinking motion is as follows:

$$m_1(s) = M(\infty) + \frac{2}{\pi}\int_0^\infty \frac{m(k)\sin ks}{k}\,dk \qquad (s > 0) \quad (2)$$

where $m(k)$ is defined

$$m(k) = M(k) - M(\infty)$$

For the case of a pitching airfoil, it was shown in reference 2 that the only additional indicial function that need be considered is the one associated with the pitching velocity of the airfoil. The reciprocal relations used in this analysis for determining the oscillatory lift and moment coefficients at the high reduced frequencies as well as the complete indicial functions for the case of pitching velocity are, for the lift,

$$k_{1_q}(s) = F_{c_q}(\infty) + \frac{2}{\pi}\int_0^\infty \frac{f_q(k)\sin ks}{k}\,dk \qquad (s > 0) \quad (3)$$

where

$$f_q = F_{c_q}(k) - F_{c_q}(\infty) \qquad (4)$$

and, for the moment,

$$m_{1_q}(s) = M_q(\infty) + \frac{2}{\pi}\int_0^\infty \frac{m_q(k)\sin ks}{k}\,dk \qquad (s > 0) \quad (5)$$

where

$$m_q(k) = M_q(k) - M_q(\infty) \qquad (6)$$

For convenience, a summary of the reciprocal relations for lift and moment due to both sinking and pitching motion is given in table I together with the definition of the functions appearing in the reciprocal relations.

AVAILABLE OSCILLATORY COEFFICIENTS AND THEIR APPROXIMATION AT HIGH REDUCED FREQUENCIES

Summary of available oscillatory coefficients at $M = 0.5$ and $M = 0.6$.- The two-dimensional compressible oscillatory coefficients for the real and imaginary parts of the lift and moment for sinking and pitching motion are given in table II for $M = 0.5$ and $M = 0.6$. The results given in table II were taken from two sources (see refs. 3 and 4) and have been converted to the form given by the equations noted in the last column of table I for the lift and moment. The range of reduced frequencies taken from each source is indicated in table II. Expressions for the end points of the oscillatory coefficients, that is, for $k = 0$ and $k = \infty$, were determined from the expressions given in references 1 and 2.

Expressions for indicial lift and moment functions in region $0 \leqq s \leqq \frac{2M}{1 + M}$.- The expressions for the indicial lift and moment functions due to sinking and pitching motion that are given in reference 6 for the region $0 \leqq s \leqq \frac{2M}{1 + M}$ have been converted to the notation used in this paper and are as follows:

$$k_1(s) = \frac{2}{\pi M}\left[1 - \frac{s}{2M}(1 - M)\right] \tag{7}$$

$$m_1(s) = -\frac{2}{\pi M}\left[\frac{1}{4} - \frac{s}{8M}(1 - M) - \frac{s^2}{16M}(2 - M)\right] \tag{8}$$

$$k_{1_q}(s) = \frac{1}{\pi M}\left[1 - \frac{s}{2M}(1 - M) + \frac{s^2}{4M}\left(1 - \frac{M}{2}\right)\right] \tag{9}$$

$$m_{1_q}(s) = -\frac{1}{\pi M}\left\{\frac{5}{12} - \frac{3}{8}\frac{s}{M}(1 - M) + \frac{s^2}{32M^2}(2 - 6M + 3M^2) + \frac{s^3}{12M^3}\left[\frac{1}{8}(1 - M)^3 + \frac{M}{2}\right]\right\} \tag{10}$$

<u>Approximation of oscillatory coefficients at high reduced frequencies</u>.- The following series was assumed as an approximate expression for $f(k)$ in the range $k_0 \leqq k \leqq \infty$:

$$f(k) = \sum_{n=2} a_n k e^{-\alpha_n (k-k_0)} \tag{11}$$

where a_n and α_n are the unknown coefficients to be determined for $f(k)$ at a given Mach number. Although several forms of series could be used to express $f(k)$, this form was found to be the most convenient for purposes of integration. Note that equation (11) satisfies the condition that

$$\lim_{k \to \infty} f(k) = \lim_{k \to \infty} \left[F_c(k) - F_c(\infty)\right] = 0 \tag{12}$$

Two boundary conditions which require the approximate solution for $f(k)$ in the region $k_0 \leqq k \leqq \infty$ to join the known value at $k = k_0$ continuously in magnitude and slope are

$$\left[\frac{f(k)}{k}\right]_{k=k_0} \equiv A \tag{13}$$

$$\left\{\frac{d\left[\frac{f(k)}{k}\right]}{dk}\right\}_{k=k_0} \equiv B \tag{14}$$

These two conditions are used to reduce the number of unknowns in equation (11). Substitution of these two conditions into equation (11) leads to the following expressions for the coefficients a_1 and a_2:

$$a_1 = \frac{B + A\alpha_2}{\alpha_2 - \alpha_1} \tag{15}$$

and

$$a_2 = \frac{\alpha_1 A + B}{\alpha_1 - \alpha_2} \tag{16}$$

If the expression for $f(k)$ given by equation (11) is substituted into the left-hand side of equation (1b) and if the values of a_1 and a_2 are replaced by the expressions given by equations (15) and (16), then two expressions from which α_1 and α_2 can be evaluated are obtained. For the case $M = 0.5$ and $M = 0.6$, the value of k_0 is unity, and the two desired expressions are

$$\frac{B + A\alpha_2}{\alpha_2 - \alpha_1} \frac{s_1 \cos s_1 + \alpha_1 \sin s_1}{\alpha_1^2 + s_1^2} + \frac{\alpha_1 A + B}{\alpha_1 - \alpha_2} \frac{s_1 \cos s_1 + \alpha_2 \sin s_1}{\alpha_2^2 + s_1^2} =$$

$$\frac{\pi}{2}\left[k_1(s_1) - F_c(\infty)\right] - \int_0^1 \frac{f(k)\sin ks_1}{k} dk \tag{17a}$$

and

$$\frac{B + A\alpha_2}{\alpha_2 - \alpha_1} \frac{s_2 \cos s_2 + \alpha_1 \sin s_2}{\alpha_1^2 + s_2^2} + \frac{\alpha_1 A + B}{\alpha_1 - \alpha_2} \frac{s_2 \cos s_2 + \alpha_2 \sin s_2}{\alpha_2^2 + s_2^2} =$$

$$\frac{\pi}{2}\left[k_1(s_2) - F_c(\infty)\right] - \int_0^1 \frac{f(k)\sin ks_2}{k} dk \tag{17b}$$

where α_1 and α_2 are numbers greater than zero, and s_1 and s_2 are greater than zero but less than or equal to $\frac{2M}{1 + M}$.

A simultaneous solution of the nonlinear equations (17a) and (17b) for the coefficients α_1 and α_2 was obtained by a graphical procedure. In some cases only an approximate solution for α_1 and α_2 could be obtained. In these instances, values for α_1 and α_2 were selected such that the magnitudes of error incurred in the indicial function were equal at the points s_1 and s_2. For the case of $M = 0.5$, values of $s_1 = 0.10\pi$ and $s_2 = 0.20\pi$ were chosen and, for $M = 0.6$, $s_1 = 0.12\pi$ and $s_2 \doteq 0.23\pi$ were chosen. Both of these values are within the limits $0 \leqq s \leqq \frac{2M}{1 + M}$ and are approximately equally spaced.

Values for the constants a_1 and a_2 could then be determined from equations (15) and (16). In table III values for the constants a_1, a_2, α_1, and α_2 which appear in the exponential approximation (eq. (11)) are summarized for the lift and moment due to both sinking and pitching motion.

RESULTS AND DISCUSSION

Determination of indicial functions.- With the coefficients in equation (11) determined for the lift and moment oscillatory coefficients due to both sinking and pitching motion, the indicial functions $k_1(s)$, $m_1(s)$, $k_{1_q}(s)$, and $m_{1_q}(s)$ were evaluated over the range $s > \frac{2M}{1 + M}$ for $M = 0.5$ and $M = 0.6$ by the methods of references 1 and 2. These results and corresponding functions calculated by the method of reference 6 for $0 \leqq s \leqq \frac{2M}{1 + M}$ are shown, respectively, in figures 1, 2, 3, and 4 together with the results for $M = 0.7$, obtained from references 1 and 2. Also included is the function for incompressible flow obtained by Wagner. The indicial lift and moment functions due to pitching velocity for the airfoil rotating about the three-quarter-chord point are shown in figures 5 and 6. For incompressible flow, the lift and moment are impulsive at $s = 0$ and constant thereafter; however, for compressible flow, the lift and moment are finite and time-dependent and approach a constant value very rapidly. Comparison of the indicial lift functions due to both sinking and pitching velocities for subsonic Mach numbers, figures 1 and 3, indicates that as the subsonic Mach number increases the lift at $s = 0$ becomes less and, for s equal to more than approximately two half-chords, the growth of lift to the steady state becomes less rapid. For the case of the moments about the quarter-chord point due to sinking and pitching motion, it is similarly seen from figures 2 and 4 that while the moments are impulsive at zero for incompressible flow the moments are finite and their values decrease with increasing Mach number for compressible flow. Although the moments for $s > 0$ are constant for incompressible flow, a time-dependent moment which rapidly approaches a constant exists for compressible flow where the rapidity of approach decreases with increasing Mach number.

Considerations for indicial lift due to penetration of a sharp-edge gust.- In references 1 and 6 it was shown that the indicial lift function $k_1(s)$ for the region $s > \frac{2M}{1 - M}$ is associated primarily with the lift due to circulation only. As a consequence, an approximation for the indicial lift function $k_2(s)$ due to penetration of a sharp-edge gust in this region can be determined on the basis of the relationship that exists between the two indicial lift functions $k_1(s)$ and $k_2(s)$ for the circulatory portion of the lift for incompressible flow. In reference 1 the function $k_2(s)$ is determined for the region $s > \frac{2M}{1 - M}$ on the basis of this relationship for $M = 0.7$. However, no information concerning the beginning portion of this function is available at present.

A method for determining an explicit expression for this function in the region $0 \leqq s \leqq \frac{2M}{1 + M}$ is possible from an extension of the analysis given in reference 6 where the boundary conditions for the perturbation velocity due to the penetration of a gust can be substituted for the perturbation velocity due to sinking motion. Details of the evaluation of the function $k_2(s)$ for this region are given in appendix A. The expression obtained for this function is relatively simple as can be seen from the following equation:

$$k_2(s) = \frac{s}{\pi\sqrt{M}} \qquad \left(0 \leqq s \leqq \frac{2M}{1 + M}\right) \tag{18}$$

where $k_2(s)$ is defined such that the lift on the airfoil is

$$L(s) = -\pi\rho cVUk_2(s) \tag{19}$$

The gust function $k_2(s)$ for the region $\frac{2M}{1 + M} \leqq s \leqq \frac{2M}{1 - M}$ can be determined from an extension of the results given in reference 6. An explicit expression cannot be easily obtained because of the complexity of the downwash regions that must be considered. Since the penetration function is known everywhere except in the region $\frac{2M}{1 + M} \leqq s \leqq \frac{2M}{1 - M}$, a reasonable estimate of the curve may be obtained by joining the ends of the two solutions at the points $\frac{2M}{1 + M}$ and $\frac{2M}{1 - M}$ by a smooth curve.

In figure 7, the indicial lift functions due to penetration of a sharp-edge gust are shown for $M = 0.5$, $M = 0.6$, and $M = 0.7$ for an extensive range of airfoil travel. Also shown is the solution for $M = 0$ obtained by Küssner. Inspection of figure 7 indicates that not only is the growth of lift to the steady state less rapid as the subsonic Mach number is increased but also the magnitude of lift in the beginning portion is less.

Approximation of indicial functions by analytic expressions.- Since the exponential function has a simple operational equivalent and a series of such functions has been found convenient for approximating $k_1(s)$ at $M = 0$ and $M = 0.7$, a finite sum of exponential terms was chosen to approximate the indicial lift and moment functions for sinking and pitching motion at $M = 0.5$ and $M = 0.6$ as follows:

$$b_0 + b_1 e^{-\beta_1 s} + b_2 e^{-\beta_2 s} + b_3 e^{-\beta_3 s} \tag{20}$$

In table IV values for these constants are given for the lift and moment due to sinking motion as well as for the lift and moment due to pitching motion about the three-quarter-chord point. The constants are noted for $M = 0.5$ and $M = 0.6$ together with those for $M = 0.7$ from reference 2. In reference 8 the constants are given for incompressible flow. Plots of these exponential approximations are shown in figures 1, 2, 5, and 6. Comparison of the approximations with the actual curves in these figures indicates good agreement.

The exponential approximation given by equation (20) was also found to be convenient for fitting the indicial lift functions $k_2(s)$ due to penetration of a sharp-edge gust shown in figure 7. In table IV the constants required for fitting the $k_2(s)$ function at $M = 0.5$, $M = 0.6$, and $M = 0.7$ for use in equation (20) are given. In reference 8 the constants for this indicial function are given for incompressible flow. Comparison of these approximations with the derived curves (see fig. 7) indicates excellent agreement.

Approximation of oscillatory coefficients by analytic expressions.- The corresponding approximate expressions for the harmonically oscillating airfoil can be found from equation (20) and an alternate form of the reciprocal equations. For the case of the lift due to sinking motion, the reciprocal equation is

$$F_c(k) + iG_c(k) = ik \int_0^\infty k_1(s) e^{-iks}\, ds \tag{21}$$

Substitution of the approximation for $k_1(s)$ given by equation (20) leads to the following expressions for $F_c(k)$ and $G_c(k)$

$$\left.\begin{aligned} F_c(k) &= b_0 + \frac{b_1 k^2}{\beta_1{}^2 + k^2} + \frac{b_2 k^2}{\beta_2{}^2 + k^2} + \frac{b_3 k^2}{\beta_3{}^2 + k^2} \\ G_c(k) &= \frac{b_1 \beta_1 k}{\beta_1{}^2 + k^2} + \frac{b_2 \beta_2 k}{\beta_2{}^2 + k^2} + \frac{b_3 \beta_3 k}{\beta_3{}^2 + k^2} \end{aligned}\right\} \qquad (22)$$

In a similar fashion, expressions for the moment functions $M(k)$ and $N(k)$ can be obtained in the form given by equation (22) as well as for the lift functions due to pitching velocity about the three-quarter-chord point $\left(F_{c_q}\right)_{3c/4}(k)$ and $\left(G_{c_q}\right)_{3c/4}(k)$ and the moment functions $\left(M_q\right)_{3c/4}(k)$ and $\left(N_q\right)_{3c/4}(k)$. The transfer of these coefficients from the leading edge to the three-quarter-chord point was made with the aid of equations (27) of reference 2.

Estimate of reliability of indicial functions.- Although the indicial functions are known exactly in the region $0 \leqq s \leqq \frac{2M}{1 + M}$, the reliability of these functions for values of $s > \frac{2M}{1 + M}$ is questionable since the oscillatory coefficients at the high reduced frequencies, which were used in determining the indicial function in this region, have been determined only approximately. An estimate of the reliability of the indicial functions in this region due to the errors in the oscillatory coefficients at the high reduced frequencies could be made on the basis of the differences which exist between the approximate solution (based on the oscillatory coefficients) and the exact solution (based on ref. 6) of the indicial functions in the region $0 \leqq s \leqq \frac{2M}{1 + M}$. Since the oscillatory coefficients at the high reduced frequencies principally affect the beginning portion of the function $k_1(s)$, the differences between the approximate and exact solutions in this region should give a reasonable and conservative estimate of the errors incurred in the indicial functions for larger values of s.

In appendix B, a chart (see fig. 8) is obtained for indicating the variation of errors in the indicial functions with airfoil travel s due to the errors in the in-phase oscillatory coefficients at the high reduced frequencies. Although figure 8 indicates the errors in only the indicial lift function $k_1(s)$, the results in this figure are also applicable for the other indicial functions. Inspection of figure 8 for $k_0 = 1$ indicates that the maximum error in the indicial function occurs within a quarter of a chord regardless of the position of the maximum error occurring in the in-phase oscillatory coefficients (as indicated by several values of κ). Since $s = 0.5$ lies within the region $0 \leqq s \leqq \frac{2M}{1 + M}$ for $M = 0.5$ and $M = 0.6$, it may be expected that the greatest errors in the indicial functions occur in this region and that errors beyond this point approach zero very rapidly. The largest differences between the indicial functions determined from the oscillatory coefficients and by the method given in reference 6 in the region $0 \leqq s \leqq \frac{2M}{1 + M}$ were found to be less than 1.5 percent for $M = 0.5$ and $M = 0.6$. Thus, it appears from figure 8 that the largest errors in the indicial functions beyond the point $s = \frac{2M}{1 + M}$ are much less than 1.5 percent and appear to approach zero very rapidly for increasing values of airfoil travel s.

Effects of discrepancies between data of reference 7 and those of present investigation on indicial functions.- Just prior to completion of this paper oscillatory lift and moment coefficients which cover a larger range of reduced frequencies than that considered by Dietze and Schade were published (see ref. 7). In figures 9 and 10 a comparison is shown of the in-phase coefficients used in the present analysis with those obtained from reference 7. For the range of reduced frequencies $0 \leqq k \leqq 2.5$, the data agree within approximately 3 percent. However, for $k > 2.5$ the differences tend to become larger than 3 percent in some cases.

Consideration of the over-all effect of the discrepancies shown in figures 9 and 10 indicates that these differences would result in a maximum error of 3 percent in the indicial functions. This amount of error does not appear serious, and in view of the few data available from reference 7, recomputation of the indicial function does not appear worthwhile. It might be noted that the data of Dietze and Schade agree better mutually than with the data of reference 7. The reason for these discrepancies is not known; consequently, the selection of the most accurate oscillatory coefficients is still questionable.

CONCLUDING REMARKS

From approximate expressions for the in-phase oscillatory coefficients determined for the high reduced frequencies and with a knowledge of the existing data at the low reduced frequencies, the indicial lift and moment functions have been determined for sinking and pitching motion at Mach numbers M of 0.5 and 0.6.

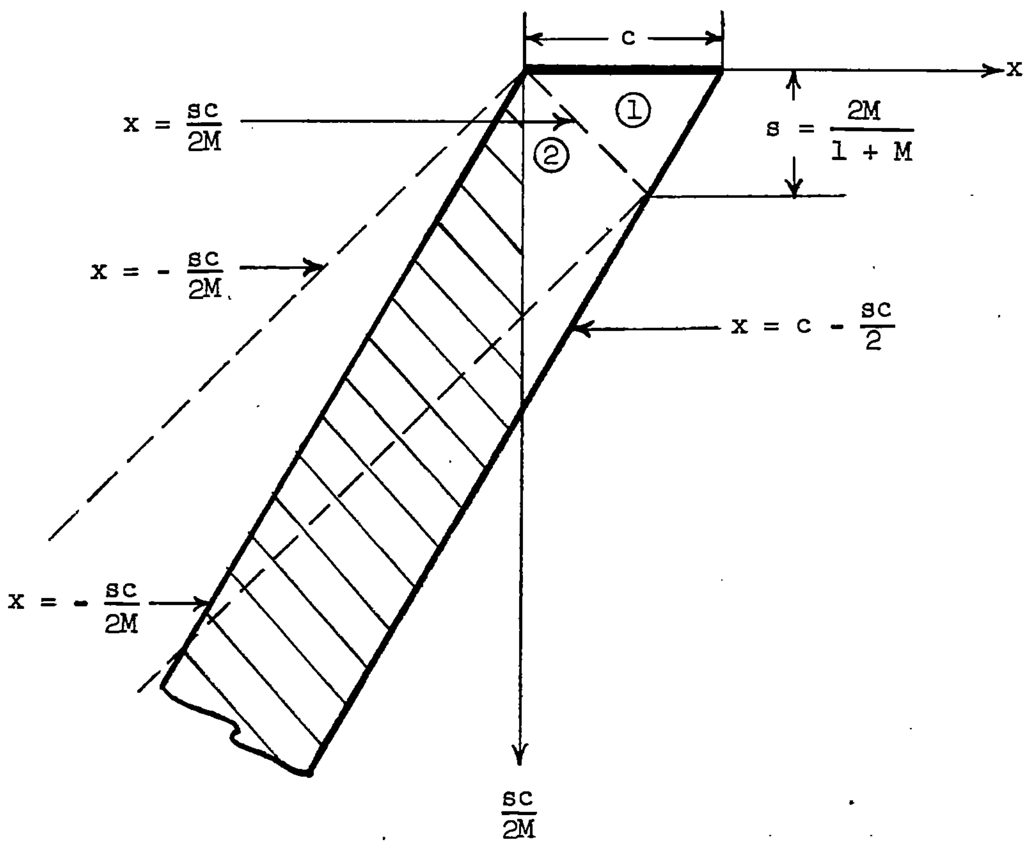
The indicial lift function due to penetration of a sharp-edge gust is determined for $M = 0.5$, $M = 0.6$, and $M = 0.7$ for an extensive range of airfoil travel. Inspection of these functions indicates that not only is the growth of lift to the steady state less rapid as the subsonic Mach number is increased but also the magnitude of lift in the beginning portion is less.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., April 15, 1952.

APPENDIX A

DETERMINATION OF INDICIAL LIFT FUNCTION $k_2(s)$ DUE TO PENETRATION OF A SHARP-EDGE GUST FOR $0 \leqq s \leqq \frac{2M}{1 + M}$

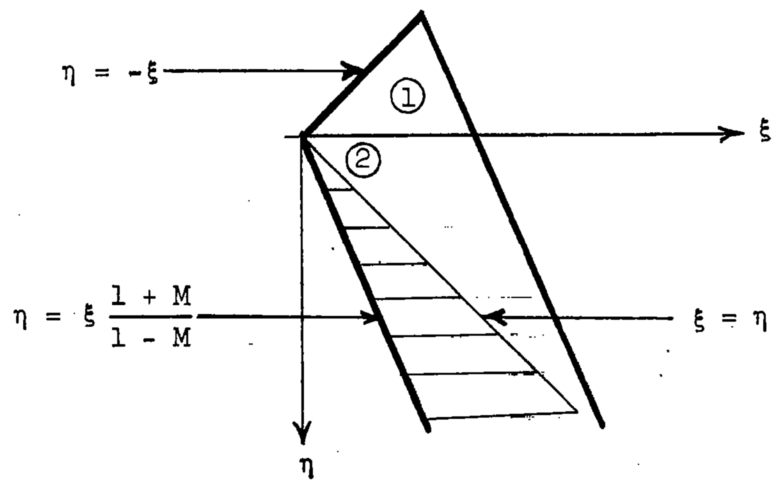
In reference 6 a method is outlined for determining the subsonic indicial lift functions for a sinking airfoil. However, if the boundary conditions for the perturbation velocity due to the penetration of a gust are substituted for the perturbation velocity due to sinking motion, the function $k_2(s)$ can be obtained. Because of the complexity of the problem arising after the Mach forecone intersects the trailing edge, the analysis is made only for the region $0 \leqq s \leqq \frac{2M}{1 + M}$. The boundary condition for the gust velocity U differs from the boundary condition of the sinking airfoil in that there is no perturbation velocity on the part of the airfoil left unshaded in the following sketch:



If as in reference 6 the axes in the preceding sketch are transformed by the relations

$$\xi = \frac{1}{\sqrt{2}}\left(\frac{sc}{2M} + x\right) \qquad ; \qquad \eta = \frac{1}{\sqrt{2}}\left(\frac{sc}{2M} - x\right)$$

then the geometry can be represented by



The expression for the pressure coefficient C_p can be written in terms of the ξ,η coordinate system as follows, where the subscripts indicate the region for which the particular equation applies:

$$C_{p_1} = 0 \tag{A1}$$

$$C_{p_2} = \frac{2}{\pi M}\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right) \int_{\eta\frac{1-M}{1+M}}^{\xi} \frac{d\xi_1}{\sqrt{\xi - \xi_1}} \int_{\xi_1}^{\eta} \frac{d\eta}{\sqrt{\eta - \eta_1}} \tag{A2}$$

Here it might be noted that equation (A2) differs from the corresponding equation in reference 6 only by the lower limit of the last integral since this integration extends over only that part of the wing which has penetrated the gust.

Evaluation of equations (A1) and (A2) gives

$$C_{p_1} = 0 \tag{A3}$$

$$C_{p_2} = \frac{8\sqrt{M}}{\pi(1 + M)}\sqrt{\frac{\frac{sc}{2M} - x}{\frac{sc}{2} + x}} \tag{A4}$$

and the expression for the indicial function $k_2(s)$ becomes

$$2\pi k_2(s) = \frac{1}{c}\int_{-sc/2}^{sc/2M} C_{p_2}\,dx + \frac{1}{c}\int_{sc/2M}^{c - \frac{sc}{2}} C_{p_1}\,dx \qquad \left(0 \leqq s \leqq \frac{2M}{1 + M}\right) \tag{A5}$$

or

$$k_2(s) = \frac{s}{\pi\sqrt{M}} \qquad \left(0 \leqq s \leqq \frac{2M}{1 + M}\right) \tag{A6}$$

APPENDIX B

INVESTIGATION OF ERRORS IN INDICIAL FUNCTION DUE TO ERRORS IN IN-PHASE OSCILLATORY COEFFICIENTS AT HIGH REDUCED FREQUENCIES

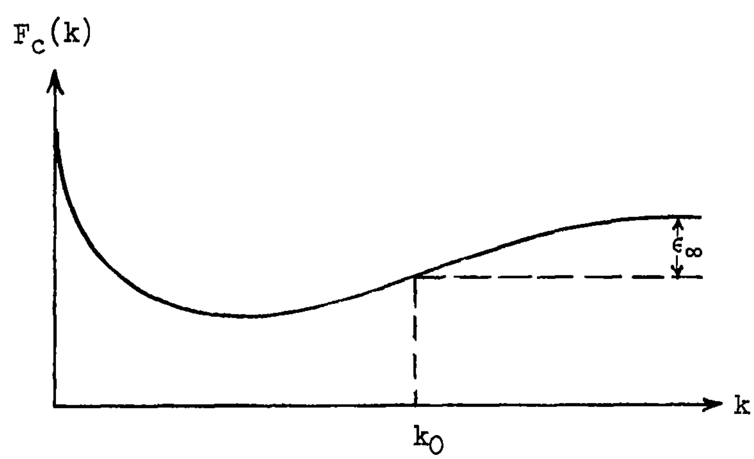
Consider the following reciprocal relation for the lift due to sinking velocity:

$$k_1(s) = \frac{2}{\pi}\int_0^\infty \frac{F_c(k)\sin ks}{k}\, dk \qquad \text{(B1)}$$

It is assumed that $F_c(k)$ is known for values of $0 \leqq k \leqq k_0$; however, for values of $k > k_0$ an error in the $F_c(k)$ function is assumed of the following form:

$$F_c(k)_{error} = \pm\epsilon_\infty\left(1 - \frac{k_0}{k}\right) \qquad \text{(B2)}$$

where ϵ_∞ is the magnitude of the error at $k = \infty$:



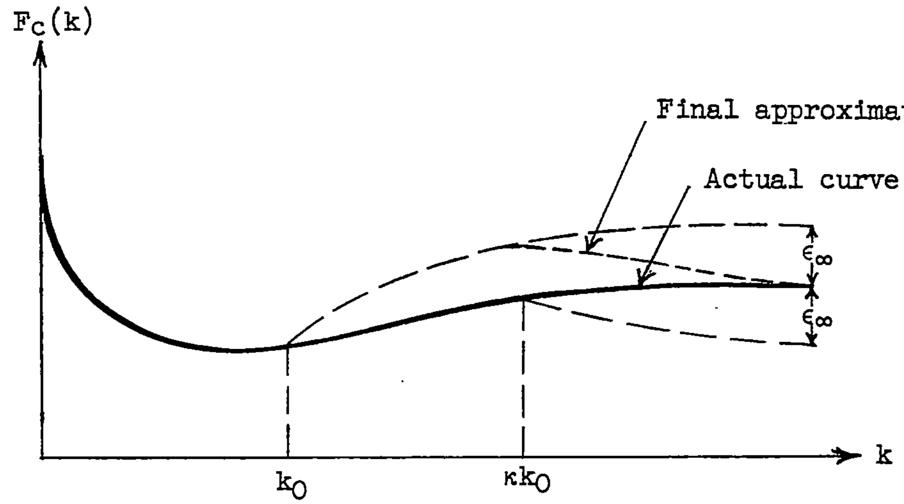
If equation (B2) is substituted into equation (B1), an expression for the error in the $k_1(s)$ function may be obtained as follows:

$$\pm \frac{\pi}{2} k_1(s)_{error} = \int_{k_0}^{\infty} \frac{\epsilon_\infty\left(1 - \frac{k_0}{k}\right)\sin ks}{k}\,dk$$

$$= \epsilon_\infty\left[\frac{\pi}{2} - \sin k_0s - Si(k_0s) + k_0s\ Ci(k_0s)\right] \tag{B3}$$

where $Si(x)$ and $Ci(x)$ are tabulated in Jahnke and Emde (ref. 9).

If the value of $F_c(\infty)$ is known and if the maximum error is assumed to occur at some arbitrary point κk_0 (where $\kappa > 1$), two error functions of the type given by equation (B2), where one is displaced from the point k_0 by an amount κk_0, may be superposed such that the value of $F_c(\infty)_{error} = 0$ as illustrated in the following sketch:



The value of the maximum error which occurs at $k = \kappa k_0$ can be expressed in terms of ϵ_∞ as follows:

$$\epsilon_{max} = \epsilon_\infty\left(1 - \frac{1}{\kappa}\right) \tag{B4}$$

The corresponding expression for the error in the function $k_1(s)$ due to the total error in the function $F_c(k)$ shown in the preceding sketch is

$$\pm \frac{\pi}{2\epsilon_{max}} k_1(s)_{error} = \frac{\kappa}{\kappa - 1}\Big[k_0 s \; Ci(k_0 s) - s\kappa k_0 \; Ci(\kappa k_0 s) + Si(\kappa k_0 s) - Si(k_0 s) + \sin(\kappa k_0 s) - \sin(k_0 s)\Big] \tag{B5}$$

A plot of equation (B5) as a function of $k_0 s$ is shown in figure 8 for several values of the parameter κ. Thus, for a maximum error ϵ_{max} in the $F_c(k)$ function at any point $k_0 < \kappa k_0 < \infty$, an estimate of the corresponding error in the function $k_1(s)$ can be obtained from figure 8.

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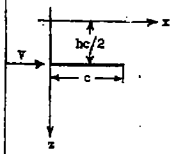
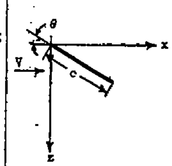
TABLE I

SUMMARY OF RECIPROCAL RELATIONS FOR LIFT AND MOMENT DUE TO SINKING AND PITCHING MOTION IN SUBSONIC FLOW

Motion	Diagram		Reciprocal relations	Definition of indicial function	Definition of oscillatory coefficient
Sinking			Lift		
			$k_1(s) = \frac{2}{\pi}\int_0^{\infty}\frac{F_c(k)\sin ks}{k}\,dk$ $k_1(s) = F_c(0) + \frac{2}{\pi}\int_0^{\infty}\frac{G_c(k)\cos ks}{k}\,dk$	$L(s) = -\pi\rho cV^2\frac{dh}{ds}k_1(s)$	$L(s) = -\pi\rho cV^2e^{iks}(-ihk)\left[F_c(k) + iG_c(k)\right]$
			Moment[1]		
			$m_1(s) = \frac{2}{\pi}\int_0^{\infty}\frac{M(k)\sin ks}{k}\,dk$ $m_1(s) = M(0) + \frac{2}{\pi}\int_0^{\infty}\frac{N(k)\cos ks}{k}\,dk$	$M(s) = \pi\rho c^2V^2\frac{dh}{ds}m_1(s)$	$M(s) = \pi\rho c^2V^2e^{iks}(-ihk)\left[M(k) + iN(k)\right]$
Pitching		Angular position	Lift		
			$k_1(s) = \frac{2}{\pi}\int_0^{\infty}\frac{F_c(k)\sin ks}{k}\,dk$ $k_1(s) = F_c(0) + \frac{2}{\pi}\int_0^{\infty}\frac{G_c(k)\cos ks}{k}\,dk$	$L(s) = -\pi\rho cV^2\theta k_1(s)$	$L(s) = -\pi\rho cV^2\theta e^{iks}\left[F_c(k) + iG_c(k)\right]$
			Moment[1]		
			$m_1(s) = \frac{2}{\pi}\int_0^{\infty}\frac{M(k)\sin ks}{k}\,dk$ $m_1(s) = M(0) + \frac{2}{\pi}\int_0^{\infty}\frac{N(k)\cos k(s)}{k}\,dk$	$M(s) = \pi\rho c^2V^2\theta m_1(s)$	$M(s) = \pi\rho c^2V^2\theta e^{iks}\left[M(k) + iN(k)\right]$
		Pitching velocity	Lift		
			$k_{1_q}(s) = \frac{2}{\pi}\int_0^{\infty}\frac{F_{c_q}(k)\sin ks}{k}\,dk$ $k_{1_q}(s) = F_{c_q}(0) + \frac{2}{\pi}\int_0^{\infty}\frac{G_{c_q}(k)\cos ks}{k}\,dk$	$L(s) = -\pi\rho cV^2 2qk_{1_q}(s)$	$L(s) = -\pi\rho cV^2 2ik\theta e^{iks}\left[F_{c_q}(k) + iG_{c_q}(k)\right]$
			Moment[1]		
			$m_{1_q}(s) = \frac{2}{\pi}\int_0^{\infty}\frac{M_q(k)\sin ks}{k}\,dk$ $m_{1_q}(s) = M_q(0) + \frac{2}{\pi}\int_0^{\infty}\frac{N_q(k)\cos ks}{k}\,dk$	$M(s) = \pi\rho c^2V^2 2qm_{1_q}(s)$	$M(s) = \pi\rho c^2V^2 2ik\theta e^{iks}\left[M_q(k) + iN_q(k)\right]$

[1]Moment taken about quarter-chord point.

TABLE II

OSCILLATORY COEFFICIENTS FOR THE LIFT AND MOMENT DUE TO SINKING AND PITCHING MOTION AT M = 0.5 AND M = 0.6

Reference	k	$F_c(k)$ for -		$-G_c(k)$ for -		$F_{c_q}(k)$ for -		$-G_{c_q}(k)$ for -		$-M(k)$ for -		$-N(k)$ for -		$-M_q(k)$ for -		$-N_q(k)$ for -	
		M = 0.5	M = 0.6	M = 0.5	M = 0.6	M = 0.5	M = 0.6	M = 0.5	M = 0.6	M = 0.5	M = 0.6	M = 0.5	M = 0.6	M = 0.5	M = 0.6	M = 0.5	M = 0.6
1 and 2	0	1.1547	1.2500	0	0	0.8660	0.9375	0	0	0	0	0	0	0.07217	0.07812	0	0
3	.02	1.0962	1.1747	.1042	.1307	.8323	.8798	.09875	.1177	.000125	.00025	.00375	.004500	.07327	.08397	.002625	.002250
	.04	1.0385	1.1010	.1517	.1885	.7777	.8239	.1191	.1496	.00050	.00100	.007375	.009125	.07292	.07890	.004825	.005875
	.06	.9858	1.0358	.1766	.2176	.7362	.7768	.1413	.1756	.0008333	.001667	.01092	.01317	.07280	.08037	.006619	.007967
	.08	.9394	.9794	.1885	.2363	.7032	.7728	.1547	.1614	.001312	.002500	.01425	.01712	.07347	.08075	.009009	.01054
	.10	.8980	.9315	.1919	.2321	.6732	.6938	.1657	.1972	.001700	.003300	.01760	.02100	.07369	.08083	.01070	.01254
	.20	.7642	.7830	.1535	.1877	.5693	.5782	.1490	.1729	.004075	.007500	.03353	.03875	.07552	.08451	.02020	.02260
	.30	.6958	.7092	.08966	.1205	.5155	.5201	.1176	.1463	.006483	.01187	.04907	.05600	.07776	.08809	.02912	.03173
	.40	.6607	.6759	.02475	.0559	.4864	.4900	.08310	.1147	.009250	.01693	.06475	.07312	.08014	.09254	.03780	.04033
	.50	.6447	.6645	-.03750	-.002200	.4708	.4756	.05249	.08808	.01260	.02324	.08050	.09020	.08310	.09768	.04652	.04850
	.60	.6415	.6672	-.09550	-.05383	.4644	.4695	.02498	.06646	.01675	.03127	.09667	.1067	.08641	.1042	.05509	.05627
	.70	.6479	.6804	-.1502	-.09836	.4644	.4690	.00041	.04881	.02200	.04150	.1130	.1244	.09047	.1105	.06357	.06346
4	.80	.6619	.6992	-.2004	-.1377	.4665	.4707	-.02102	.03683	.02737	.05116	.1293	.1409	.09385	.1165	.07192	.06975
	1.00	.7105	.7636	-.2928	-.1890	.4861	.4839	-.05603	.02883	.04405	.08590	.1643	.1727	.1057	.1364	.08884	.07806
1 and 2	∞	1.2732	1.0610	0	0	.6366	.5305	0	0	.3183	.2653	0	0	.2653	.2210	0	0

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TABLE III

COEFFICIENTS IN EXPONENTIAL APPROXIMATION $a_1ke^{-\alpha_1(k-1)} + a_2ke^{-\alpha_2(k-1)}$ FOR

OSCILLATORY LIFT AND MOMENT FUNCTIONS AT HIGH REDUCED

FREQUENCIES FOR $M = 0.5$ AND $M = 0.6$

Oscillatory function	a_1 for -		a_2 for -		α_1 for -		α_2 for -	
	$M = 0.5$	$M = 0.6$	$M = 0.5$	$M = 0.6$	$M = 0.5$	$M = 0.6$	$M = 0.5$	$M = 0.6$
$f(k)$	-0.003203	-0.1566	0.5595	0.14075	0.200	1.000	1.67	3.150
$m(k)$	-.07520	-.02003	.3495	.1994	.385	.001	1.256	2.056
$f_q(k)$	-.06812	0	.08248	.04665	.584	∞	3.155	1.100
$m_q(k)$	.11575	.08110	.04375	.003572	1.175	1.400	2.560	26.000

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TABLE IV

COEFFICIENTS IN EXPRESSION $b_0 + b_1 e^{-\beta_1 s} + b_2 e^{-\beta_2 s} + b_3 e^{-\beta_3 s}$ FOR APPROXIMATING

THE INDICIAL LIFT AND MOMENT FUNCTIONS AT M = 0.5, 0.6, AND 0.7

Indicial function	Mach number	b_0	b_1	b_2	b_3	β_1	β_2	β_3
$k_1(s)$	0.5	1.155	-0.406	-0.249	0.773	0.0754	0.372	1.890
	.6	1.250	-.452	-.630	.893	.0646	.481	.958
	.7	1.400	-.5096	-.567	-.5866	.0536	.357	.902
$m_1(s)$	.5	0	.0557	-1.000	.6263	2.555	3.308	6.09
	.6	0	-.100	-1.502	1.336	1.035	4.040	5.022
	.7	0	-.2425	.084	-.069	.974	.668	.438
$\left(k_{1_q}\right)_{3c/4}(s)$	.5	0	0	-2.68	2.362	------	4.08	4.90
	.6	0	0	0	-.2653	------	------	1.345
	.7	0	-.083	-.293	.149	.800	1.565	2.44
$\left(m_{1_q}\right)_{3c/4}(s)$	.5	-.0721	-.248	.522	-.2879	1.562	2.348	6.605
	.6	-.0781	-.077	.380	-.2469	.551	2.117	4.138
	.7	-.0875	-.00998	.1079	-.02920	.1865	1.141	4.04
$k_2(s)$	.5	1.155	-.450	-.470	-.235	.0716	.374	2.165
	.6	1.250	-.410	-.538	-.302	.0545	.257	1.461
	.7	1.400	-.563	-.645	-.192	.0542	.3125	1.474

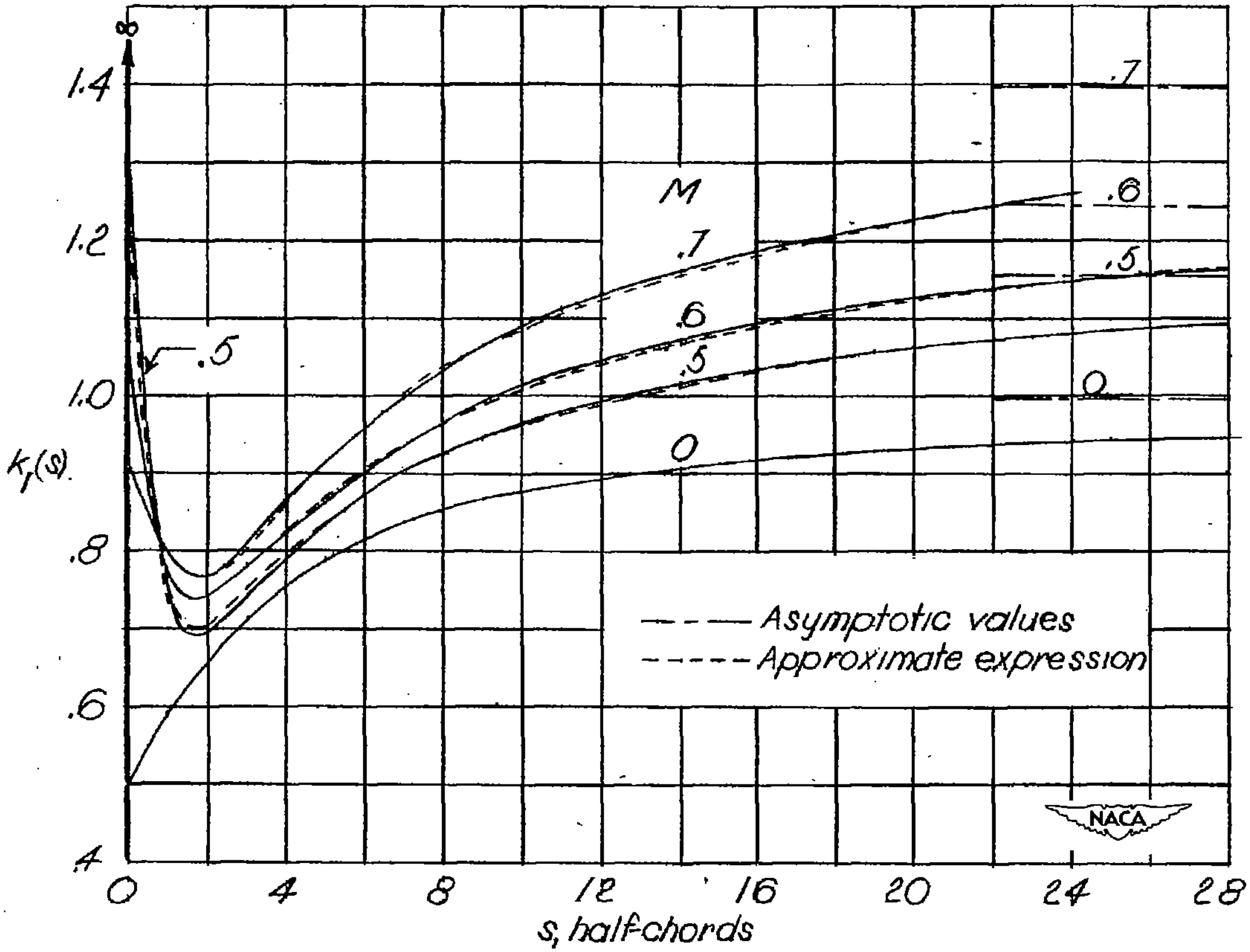


Figure 1.- Comparison of indicial lift functions due to a sudden change in sinking velocity or pitching displacement for several subsonic Mach numbers.

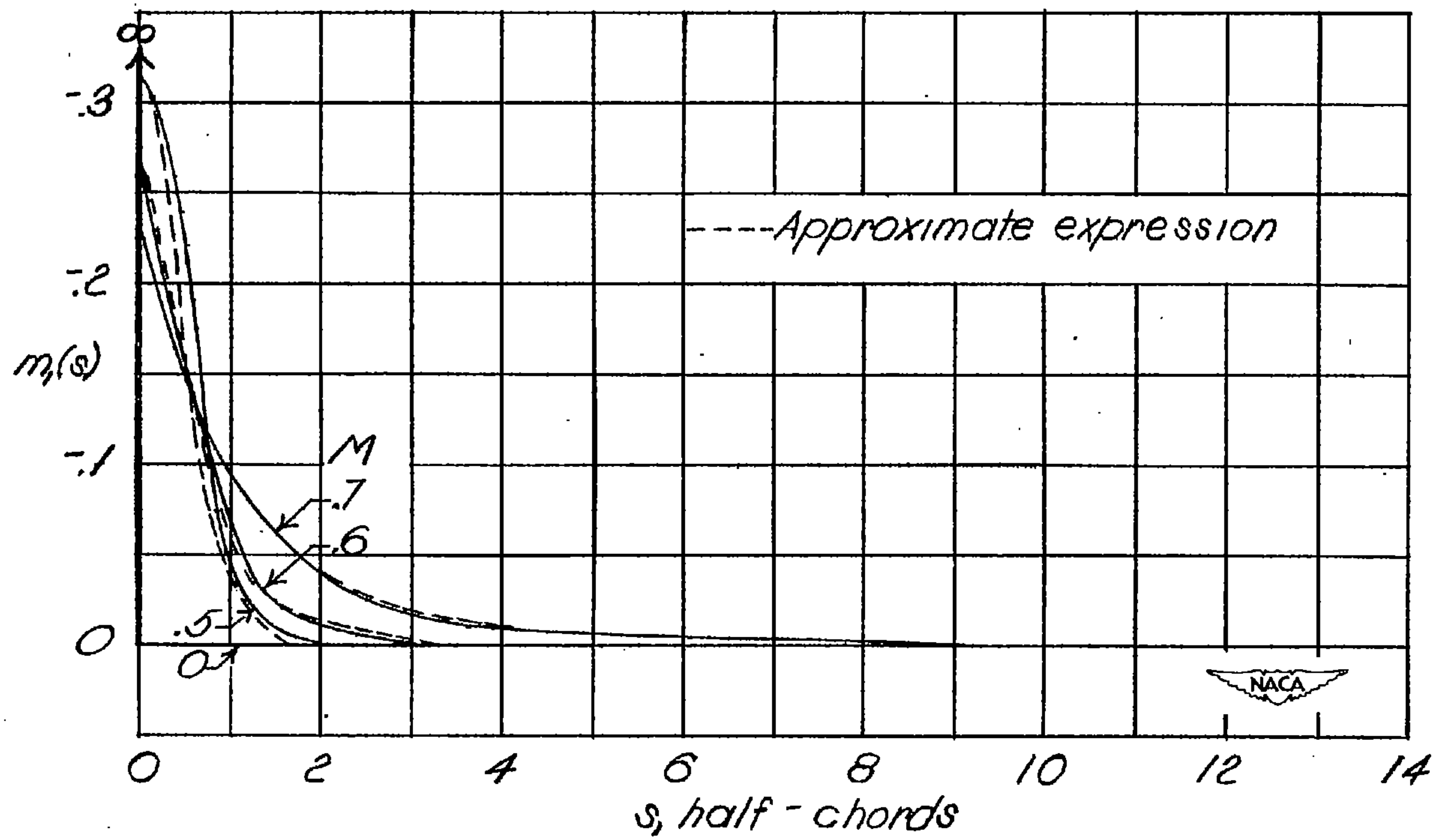


Figure 2.- Comparison of indicial moment functions due to a sudden change in sinking velocity or pitching displacement for several subsonic Mach numbers. (Moment taken about quarter-chord point.)

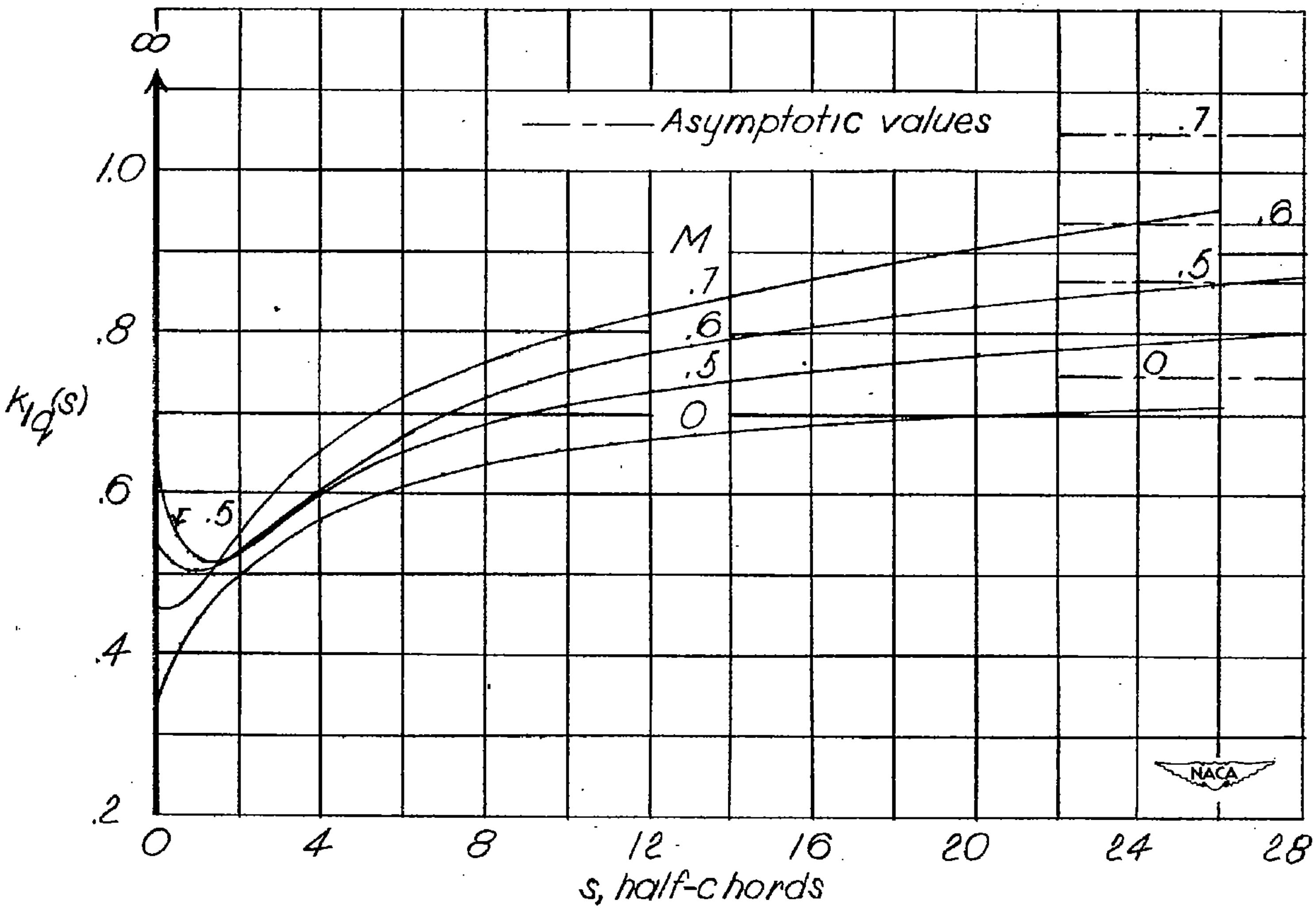


Figure 3.- Comparison of indicial lift functions due to a sudden change in pitching velocity for an airfoil rotating about its leading edge for several subsonic Mach numbers.

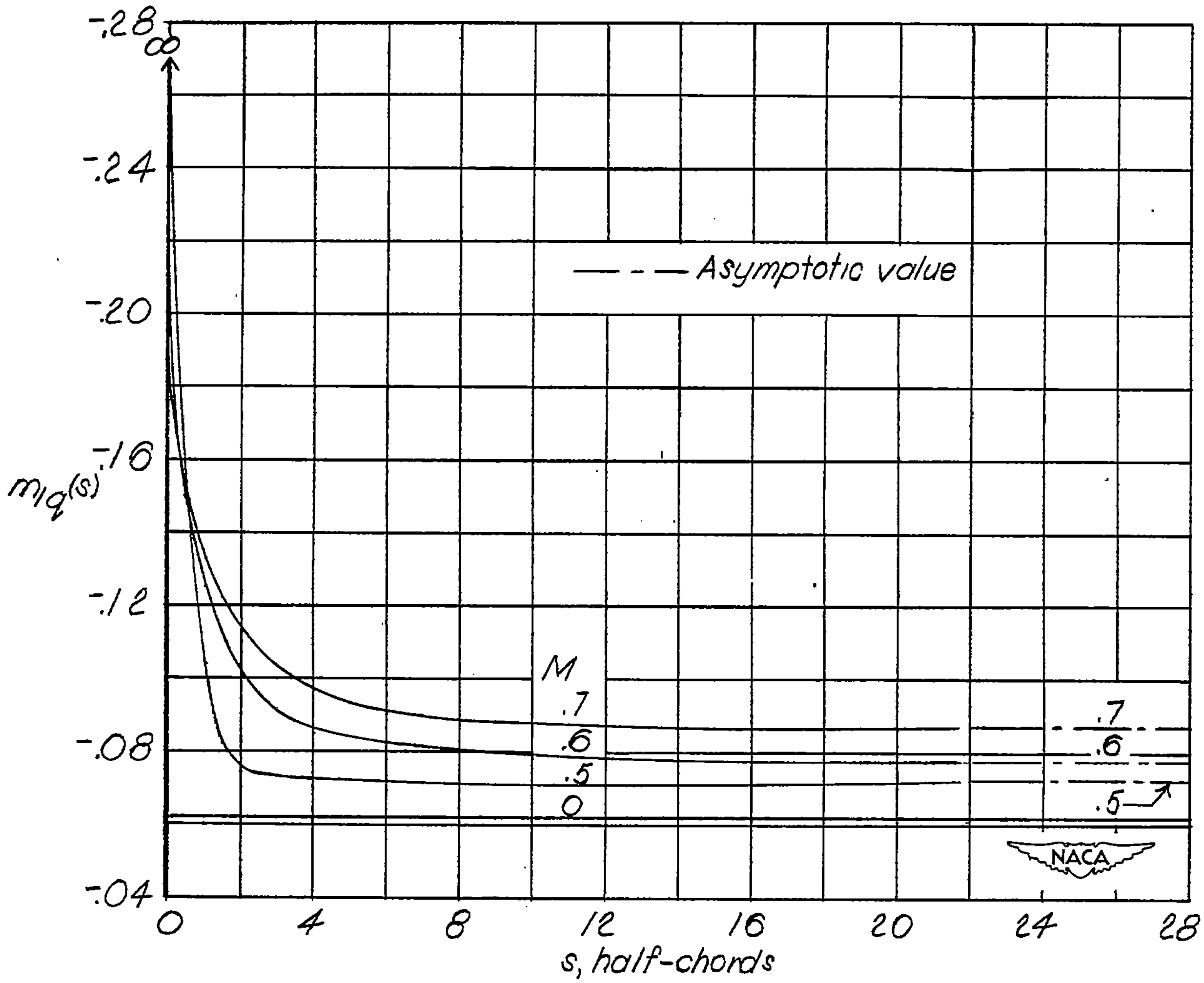


Figure 4.- Comparison of indicial moment functions due to a sudden change in pitching velocity for an airfoil rotating about its leading edge for several subsonic Mach numbers. (Moment taken about quarter-chord point.)

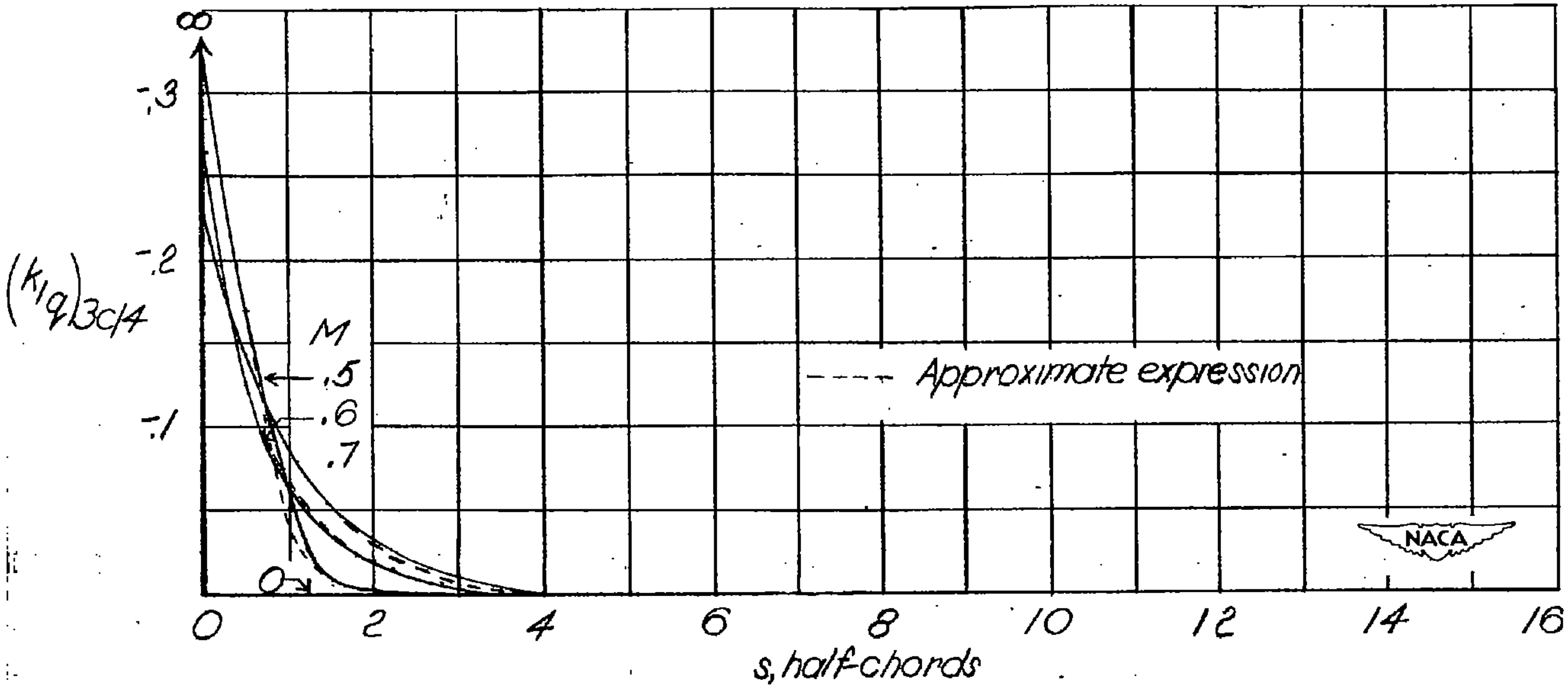


Figure 5.- Comparison of indicial lift functions due to a sudden change in pitching velocity for an airfoil rotating about its three-quarter-chord point for several subsonic Mach numbers.

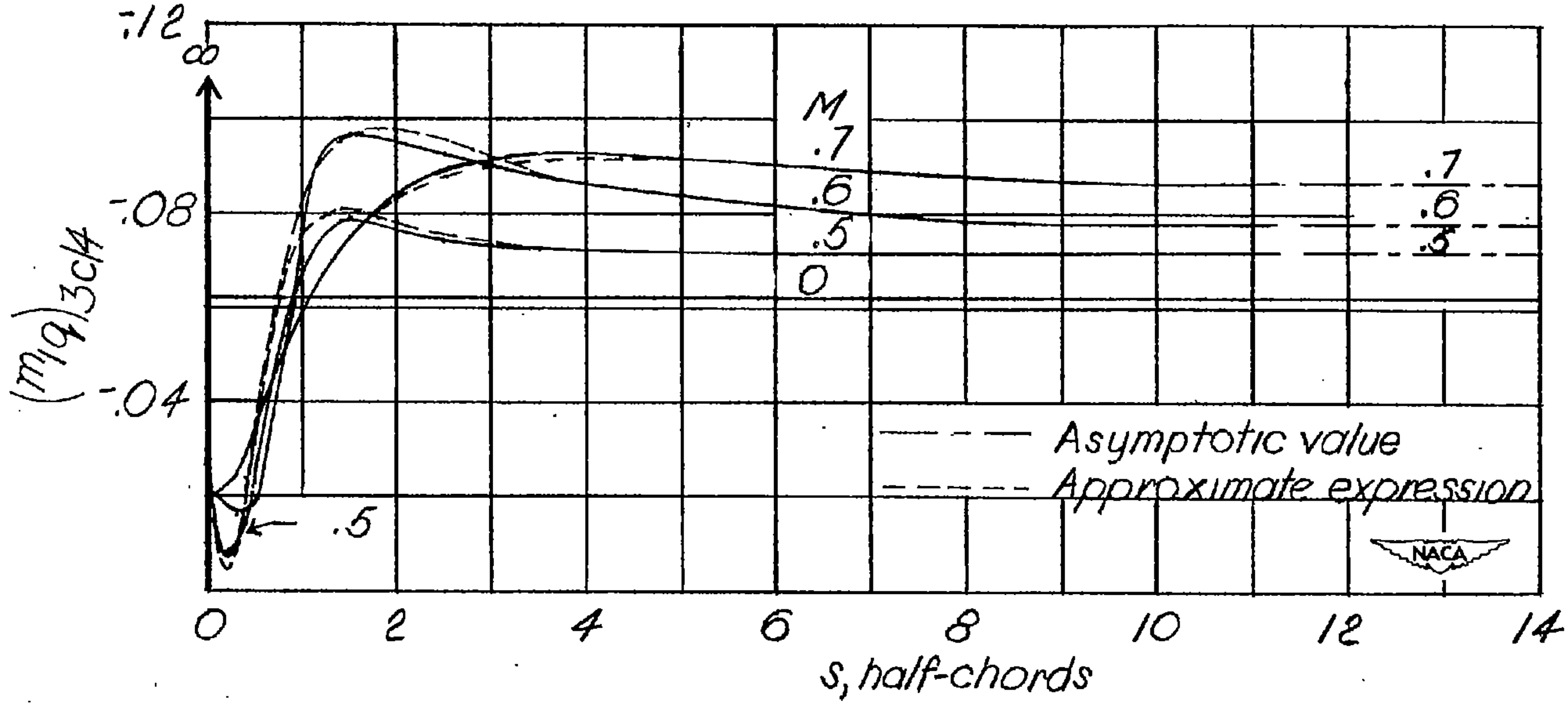


Figure 6.- Comparison of indicial moment functions due to a sudden change in pitching velocity for an airfoil rotating about its three-quarter-chord point for several subsonic Mach numbers. (Moment taken about quarter-chord point.)

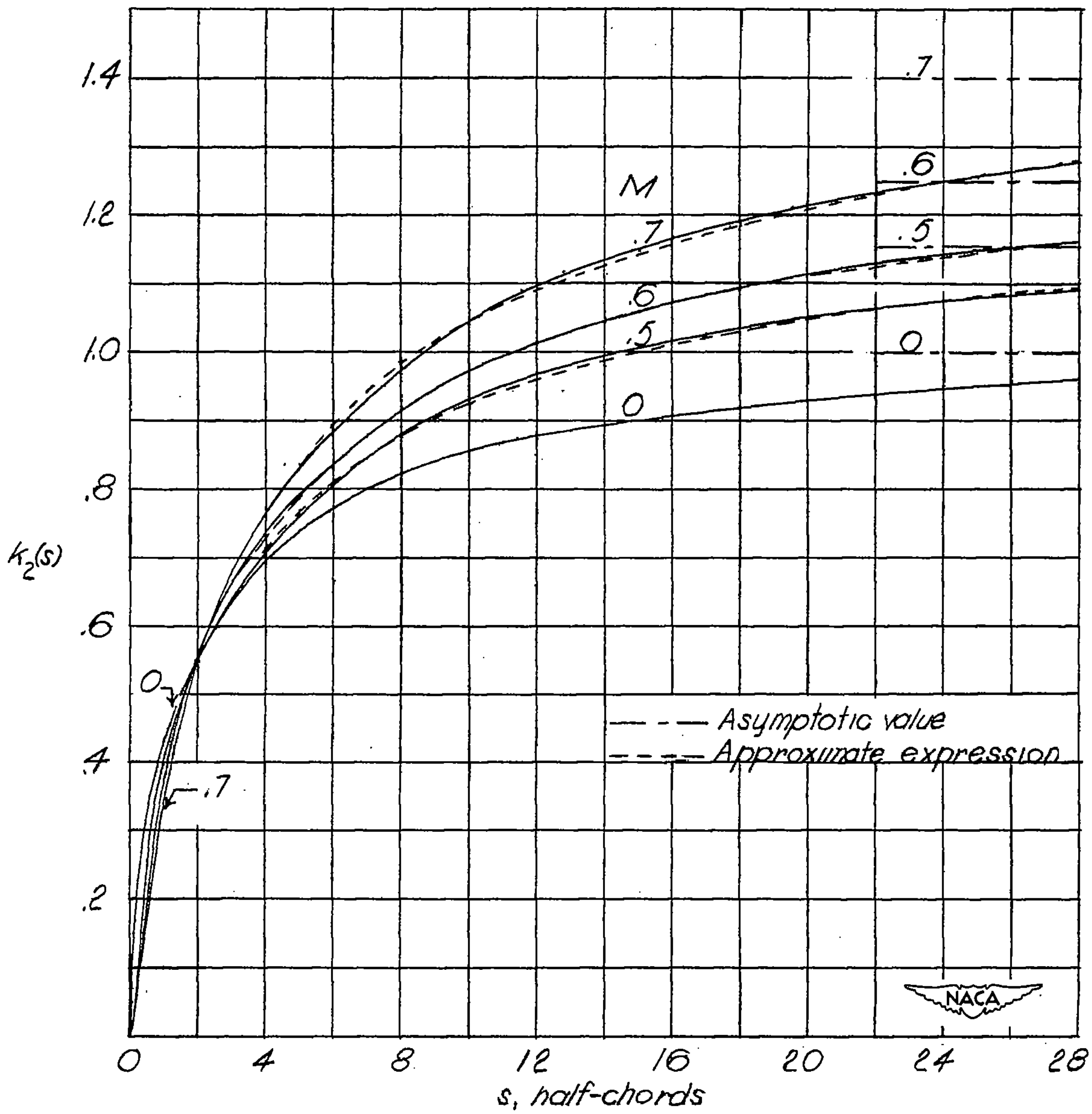


Figure 7.- Comparison of the indicial lift functions due to penetration of a sharp-edge gust at several subsonic Mach numbers.

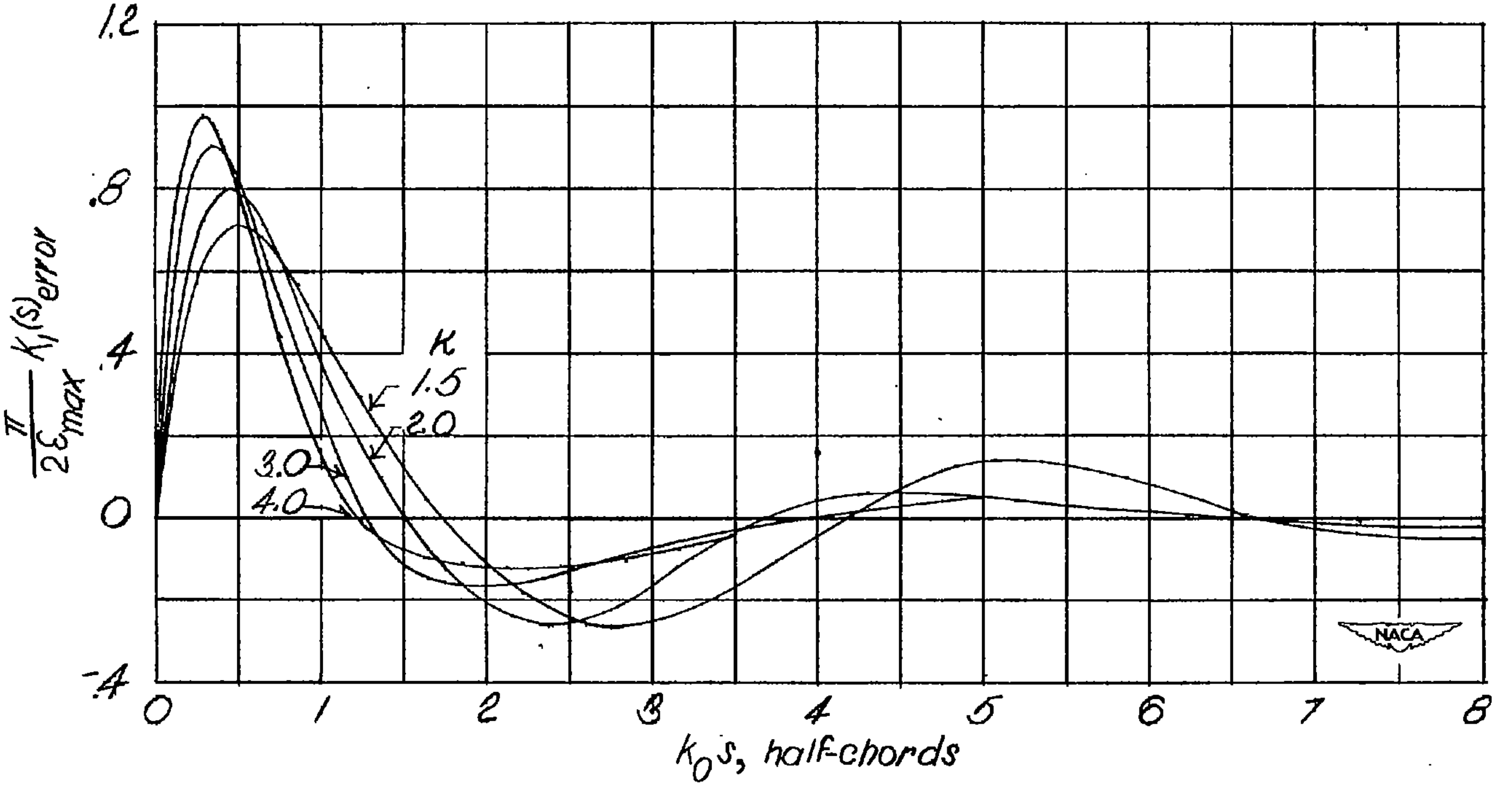
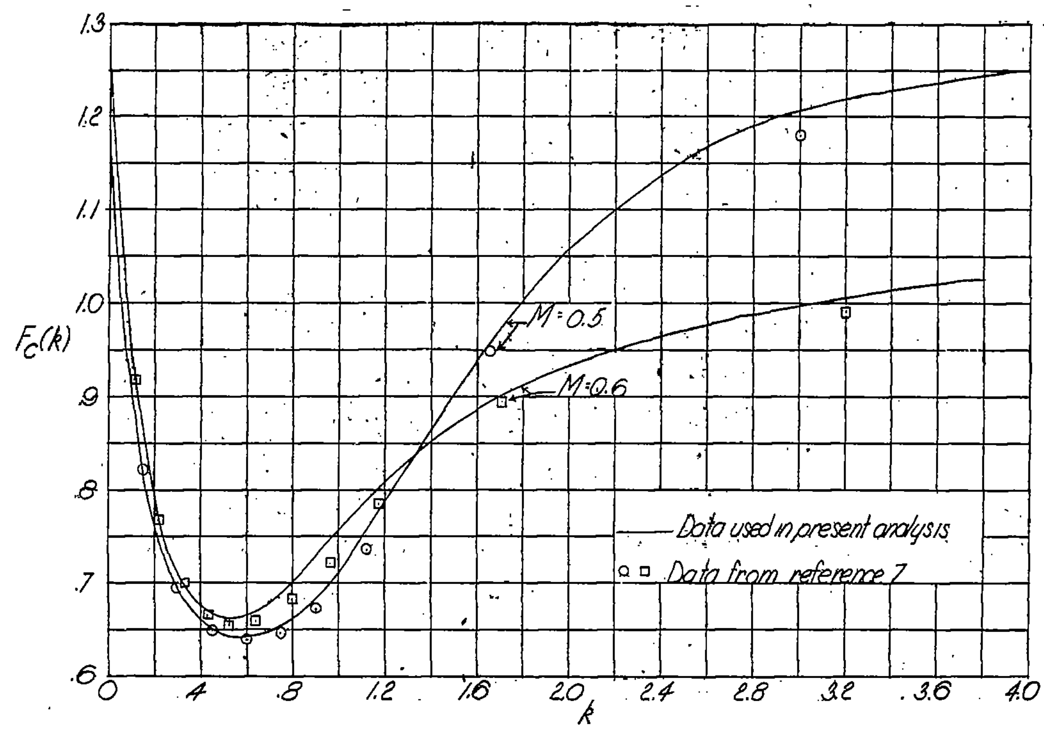
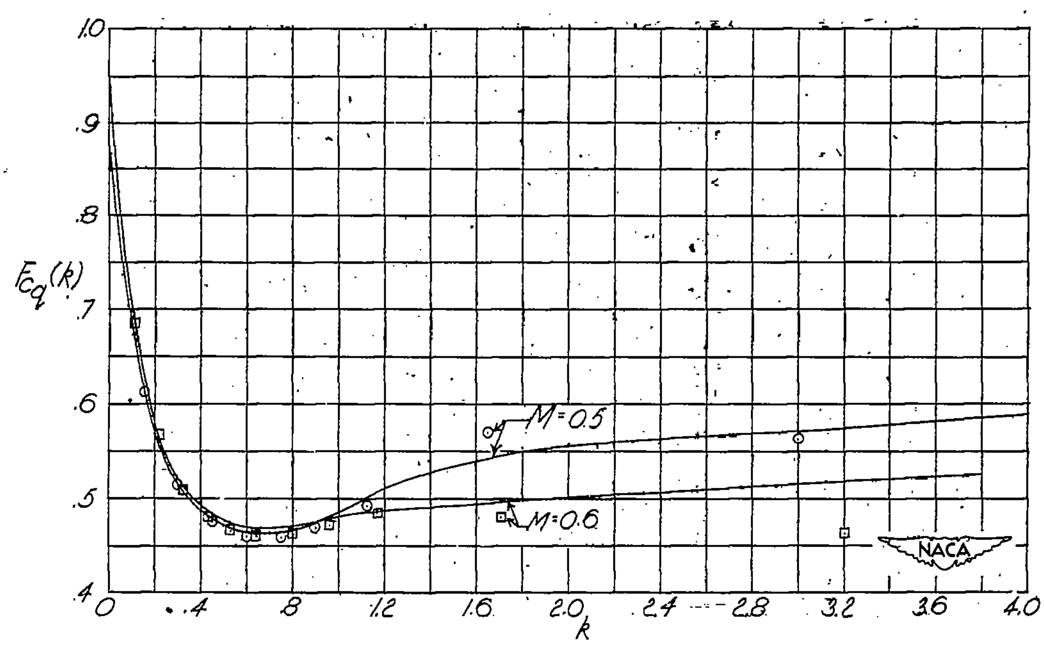


Figure 8.- Effect of errors in $f(K)$ at high reduced frequencies on indicial lift function $k_1(s)$.

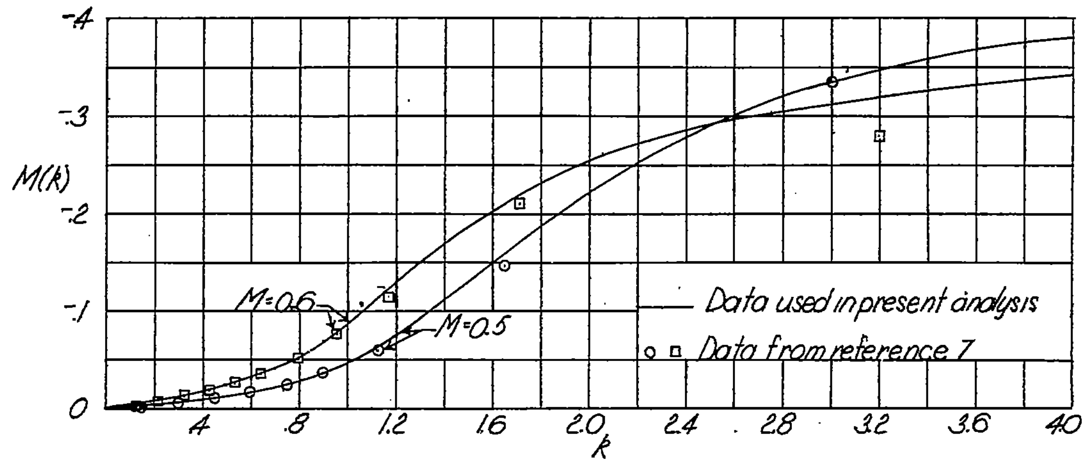


(a) Sinking velocity.

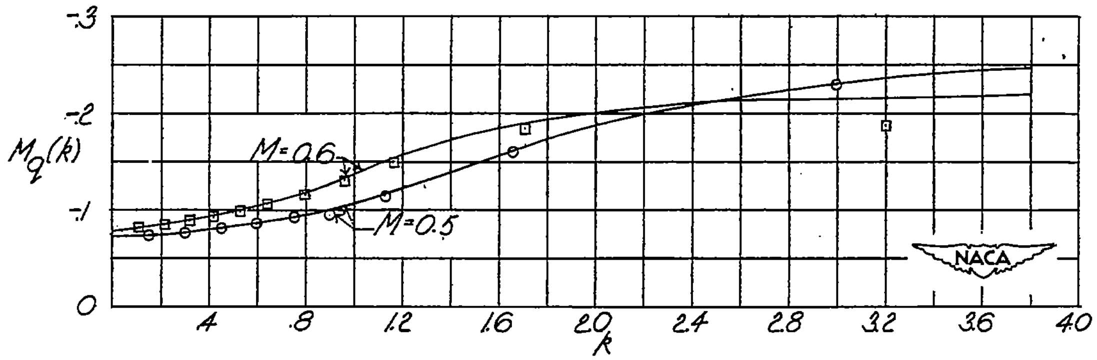


(b) Pitching velocity.

Figure 9.- Comparison of in-phase oscillatory lift coefficients used in present analysis with coefficients determined from reference 7.



(a) Sinking velocity.



(b) Pitching velocity.

Figure 10.- Comparison of in-phase oscillatory moment coefficients used in present analysis with coefficients determined from reference 7.